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Towards a strong coupling theory for the KPZ equation

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Abstract

After a brief introduction we review the nonperturbative weak noise approach to the KPZ equation in one dimension. We argue that the strong coupling aspects of the KPZ equation are related to the existence of localized propagating domain walls or solitons representing the growth modes; the statistical weight of the excitations is governed by a dynamical action representing the generalization of the Boltzmann factor to kinetics. This picture is not limited to one dimension. We thus attempt a generalization to higher dimensions where the strong coupling aspects presumably are associated with a cellular network of domain walls. Based on this picture we present arguments for the Wolf-Kertez expression $z = (2d+1)/(d+1)$ for the dynamical exponent.

Keywords: kinetics, nonequilibrium growth, growing interface, strong coupling, dynamical scaling, scaling exponents, solitons, domain walls, WKB, Langevin equation, Fokker-Planck equation, field equations, morphology, pattern formation, steps, cellular network

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1 Introduction

Driven systems far from equilibrium constitute an enormous class of natural phenomena including turbulence in fluids, interface and growth problems, chemical reactions, processes in glasses and amorphous systems, biological processes, and even aspects of economical and sociological structures.

In recent years much of the focus of modern statistical physics and soft condensed matter has shifted towards such systems. Drawing on the case of static

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and dynamic critical phenomena in and close to equilibrium, where scaling and the concept of universality have successfully served to organize our understanding and to provide a variety of calculational tools, a similar strategy has been advanced towards the much larger class of nonequilibrium phenomena with the purpose of elucidating scaling properties and more generally the morphology or pattern formation in a driven nonequilibrium state.

In this context the Kardar-Parisi-Zhang (KPZ) equation, describing the nonequilibrium growth of a noise-driven interface, provides a simple continuum model of an open driven nonlinear system exhibiting scaling and pattern formation [1]. The KPZ equation for the time evolution of the height $h(\vec{x},t)$ has the form [2]

$$\frac{\partial h(\vec{x},t)}{\partial t} = \nu\nabla^2 h(\vec{x},t) + \frac{\lambda}{2}\vec{\nabla}h(\vec{x},t)\cdot\vec{\nabla}h(\vec{x},t) + \eta(\vec{x},t) - F \ , \tag{1}$$

$$\langle\eta(\vec{x},t)\eta(\vec{x}',t')\rangle = \Delta\delta(\vec{x}-\vec{x}')\delta(t-t') \ . \tag{2}$$

Here ν characterizes the damping term, λ controls the growth term, F is a general drift term, and $\eta(\vec{x},t)$ a locally correlated white Gaussian noise modelling the stochastic nature of the drive or environment; the noise correlations are characterized by the noise strength Δ.

Starting from a given initial interface the noise drives after a transient period the interface into a stationary fluctuating state. Choosing a co-moving frame by setting $F = (\lambda/2)\langle\vec{\nabla}h\cdot\vec{\nabla}h\rangle$ the mean height $\langle h\rangle$ decays to zero but the fluctuation spectrum about $\langle h\rangle = 0$, characterized by the moments $\langle h^n\rangle$ and generally by the distribution $P(\{h(x)\},t)$, assumes a nontrivial and largely unknown form. In terms of the local slope $\vec{u} = \vec{\nabla}h$ the KPZ equation assumes the form of the Burgers equation driven by conserved noise $\left(\partial/\partial t - (\vec{u}\cdot\vec{\nabla})\right)\vec{u} = \nu\nabla^2\vec{u} + \vec{\nabla}\eta$ which within the framework of perturbative dynamic renormalization group (DRG) theory was first studied as a model of noise-driven irrotational fluid flow [3]. This analysis was reviewed and extended considerably in the context of the KPZ equation [2]. The DRG analysis basically addresses the long time - large distance scaling properties as illustrated for example by the dynamic scaling hypothesis [1] applied to the stationary height-height correlations

$$\langle(h(\vec{x},t) - h(\vec{x}',t'))^2\rangle = |\vec{x}-\vec{x}'|^{2\zeta}G(|t-t'|/|\vec{x}-\vec{x}'|^z) \ . \tag{3}$$

Here the roughness exponent ζ, the dynamic exponent z, and the scaling function G together define the KPZ universality class. Power counting or a DRG analysis identifies $d = 2$ as the lower critical dimension. An ϵ-expansion about $d = 2$ yields a kinetic phase transition at a finite growth strength λ above $d = 2$ from a weak coupling smooth phase with $z = 2$ and $\zeta = 0$ to a strong coupling rough phase with largely unknown scaling exponents.

In $d = 1$ the DRG analysis fortuitously yields $z = 3/2$ and $\zeta = 1/2$. This result also follows from the Galilean invariance of (1), $\vec{x} \to \vec{x} - \lambda\vec{u}_0 t$, $\vec{u} \to \vec{u} + \vec{u}_0$, $h \to h + \vec{u}_0\cdot\vec{x}$, which since λ is unrenormalized under the DRG, implies the scaling law $z + \zeta = 2$. Together with the known stationary distribution, $P_{\mathrm{st}} =$

$\exp(-(\nu/\Delta)\int dx\ u^2)$ [1], of independent slope fluctuations, yielding $\zeta = 1/2$, we then obtain $z = 3/2$. However, unlike critical phenomena, the DRG analysis even carried to orders beyond first loop order [4] fails to yield detailed insight into the scaling and pattern formation mechanism in the KPZ equation.

A mode coupling analysis (MC) of the KPZ equation has also been carried out both with regard to a determination of the scaling function G in $d = 1$ and in search for an elusive upper critical dimension assumed to be at $d = 4$ beyond which $z = 2$ and $\zeta = 0$ [5]. Of other approaches to the scaling properties associated with the KPZ universality class we mention briefly i) the mapping of the KPZ equation to the problem of directed polymers in a quenched random environment [1], a problem in ill-condensed matter, ii) exact results obtained for the $d = 1$ asymmetric exclusion model [6] which is in the same universality class as the KPZ equation, and finally iii) recent exact results for a discrete $d = 1$ polynuclear growth model in the KPZ universality class through the association with random matrix theory [7]. In Fig. 1 we have in a plot of the renormalized coupling strength versus the dimension summarized the scaling properties of the KPZ equation.

In summary, the KPZ equation as a continuum model of an intrinsic nonequilibrium problem and its relationship to problems in discrete driven lattice gases, random systems of the spin glass type, and random matrix theory, is of broad and paradigmatic interest.

2 Strong coupling theory

In recent work we have proposed yet another approach to the scaling and pattern formation of the KPZ equation [8] which we review below. The strong coupling approach is developed by focusing on the Fokker-Planck equation

$$\Delta\frac{\partial P(h(\vec{x}),t)}{\partial t} = HP(h(\vec{x}),t)\ ,\quad H = \int d\vec{x}\ \mathcal{H}(\vec{x})\ , \tag{4}$$

associated with the KPZ equation (1). Here the Hamiltonian or Liouvillian H drives the transition probability $P(h(\vec{x},t))$ for the height profile $h(\vec{x})$ at time t. The Fokker-Planck equation has the form of a Schrödinger equation in imaginary time for the real positive "wave function" $P(h(\vec{x}),t)$. The noise strength Δ plays the role of an effective "Planck constant". Since the KPZ equation is a stochastic field equation the associated Fokker-Planck equation is a vastly more complicated functional-differential equation. However, applying the WKB approximation valid in the "semiclassical" weak noise limit $\Delta \to 0$ by setting (h_1 in the initial height configuration)

$$P[h_1(\vec{x}) \to h(\vec{x}),t)] \propto \exp\left[-\frac{S(h_1(\vec{x}) \to h(\vec{x}),t)}{\Delta}\right]\ , \tag{5}$$

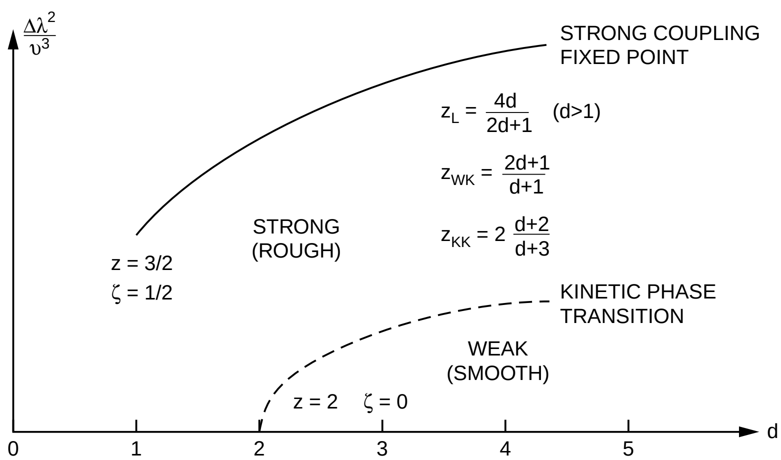

Figure 1: We summarize the scaling properties of the KPZ equation in a plot of the renormalized coupling constant (the fixed point value) $\Delta\lambda^2/\nu^3$ versus the dimension d of the system. Below $d = 2$ the scaling properties are determined by the strong coupling fixed. Above $d = 2$ the system exhibits a kinetic phase transition from a weak coupling phase to a strong coupling phase. We have also indicated three conjectures for the exponent z: The operator expansion conjecture z_L by Lässig [9] and the conjectures based on numerics, z_{WK} by Wolf and Kertész [11] and z_{KK} by Kim and Kosterlitz [10].

a "principle of least action" becomes applicable and the "weight function" or action takes the form,

$$S(h_1(\vec{x}) \to h(\vec{x}), t) = \int_{h_1(\vec{x}), t=0}^{h(\vec{x}), t} d\vec{x}dt \left[p(\vec{x}, t) \frac{\partial h(\vec{x}, t)}{\partial t} - \mathcal{H}(\vec{x}) \right] , \tag{6}$$

This implies the "classical" Hamilton equations of motion

$$\frac{\partial h(\vec{x}, t)}{\partial t} = \nu \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} \vec{\nabla} h(\vec{x}, t) \cdot \vec{\nabla} h(\vec{x}, t) - F + p(\vec{x}, t) , \tag{7}$$

$$\frac{\partial p(\vec{x}, t)}{\partial t} = -\nu \nabla^2 p(\vec{x}, t) + \lambda \vec{\nabla} p(\vec{x}, t) \cdot \vec{\nabla} h(\vec{x}, t) + \lambda p(\vec{x}, t) \nabla^2 h(\vec{x}, t) , \tag{8}$$

derived from the Hamiltonian density

$$\mathcal{H} = p(\vec{x}, t) \left[\nu \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} \vec{\nabla} h(\vec{x}, t) \cdot \vec{\nabla} h(\vec{x}, t) - F + \frac{1}{2} p(\vec{x}, t) \right] . \tag{9}$$

The field equation (7) and (8) are of the diffusive-advective type and replace the KPZ equation. The deterministic "noise field" $p(\vec{x}, t)$ corresponds to the stochastic noise $\eta(\vec{x}, t)$ in (1).

The prescription for the analysis of the KPZ equation is now straightforward. In order to determine the transition probability from an initial height profile $h_1(\vec{x})$ at time $t = 0$ to a final height profile $h(\vec{x})$ at time t we must evaluate the "classical" action associated with a "classical" orbit from $h_1(\vec{x})$ to $h(\vec{x})$ traversed in time t. The strong coupling features enters in the determination of the relevant orbits which should be nonperturbative solutions of the field equations. As is well-known from the WKB approximation in quantum theory and quantum field theory, the weak noise limit has no direct bearing on the coupling strength. The action in the present nonequilibrium context plays the same role as the energy in the Boltzmann factor in equilibrium.

Since the "classical" system is conserved the orbits are confined to lie on energy surfaces $H = \text{const}$. Here the zero energy surface $H = 0$ plays a central role in determining the stationary distribution $P_{\text{st}}(h) = \lim_{t\to\infty} P(h_1 \to h, t)$. Unlike the case in mechanical systems, the zero energy surface is unbounded and has a characteristic submanifold structure reflecting the underlying stochastic nature of the KPZ equation. The transient submanifold for $p = 0$ corresponds to the damped noiseless KPZ equation for $\eta = 0$, whereas the in general unknown stationary submanifolds is "orthogonal" to $p = 0$ in the sense that the $\int d\vec{x}\mathcal{H} = 0$ with $\mathcal{H}$ given by (9). The submanifolds intersect in the saddle point $(h, p) = (0, 0)$ corresponding to ergodic behavior. At long times a specific orbit is unbounded; however, an orbit from h_1 to h in time t is finite. At long times the orbit migrates from the finite energy surface to the zero energy submanifolds. After an initial transient period the noise drives the orbit close to the saddle point. Ergodic behavior is established and the orbit moves out along the stationary submanifold. At long times the orbit converges to the stationary submanifold and the stationary stochastic state is attained. This generic phase space behavior is depicted schematically in Fig. 2.

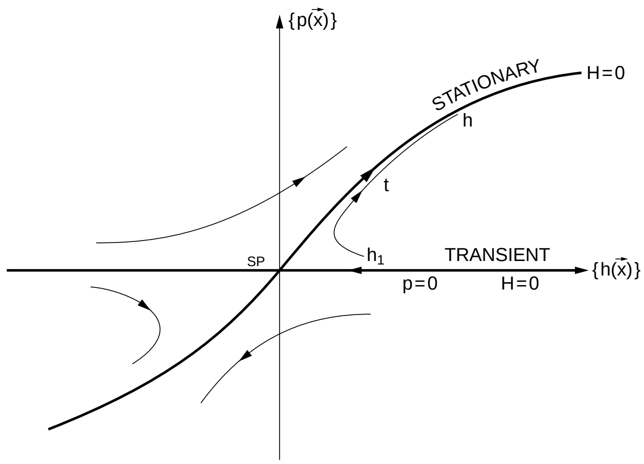

Figure 2: We depict the generic phase space behavior of the WKB approximation applied to the Fokker-Planck equation. The stationary saddle point (SP) is at the origin. The transient zero energy submanifold and the stationary zero energy submanifold are indicated. At long times the orbit from h_1 to h in time t converges to the submanifolds and passes close to the saddle point.

3 Morphology and scaling in one dimension

The strong coupling WKB analysis becomes relatively simple in one dimension [8]. The stationary zero energy submanifold is given by $p(x,t) = -2\nu\nabla^2 h(x,t)$, yielding the stationary distribution [1] $P_{\text{st}}(h) \propto \exp[-(\nu/\Delta)\int dx (dh/dx)^2]$. Moreover, the field equations (7) and (8) admit simple cusp solutions, in the static case of the form $h(x) = (2\nu/\lambda)\log|\cosh((\lambda/2\nu)u(x-x_0))|$ centered about x_0 and for large x approaching the constant slope form $h \to u|x-x_0|$. In terms of the local slope dh/dx the cusps correspond to solitons or domain walls of the kink-like form $dh/dx = u\tanh((\lambda/2\nu)|u|(x-x_0))$. The right hand soliton for $u > 0$ moves on the $p = 0$ submanifold and is the well-known viscosity-smoothed shock wave solution of the noiseless Burgers equation $(\partial/\partial t - \lambda u\nabla)u = \nu\nabla^2 u$. The left hand soliton for $u < 0$ moves on the $p = -2\nu\nabla^2 h$ submanifold and is a solution of the growing Burgers equation $(\partial/\partial t - \lambda u\nabla)u = -\nu\nabla^2 u$ with negative damping constant. The cusps or solitons form the elementary excitations or growth modes in a heuristic many body description of a growing interface. Boosting the cusps or solitons to a finite propagation velocity v by means of the Galilean transformation above we obtain the kinematic condition $v = -(2/\lambda)(u_+ + u_-)$, where u_+ and u_- are the right and left boundary values of the solitons, respectively. Matching a set of right and left hand solitons according to the kinematic condition we thus obtain a many body or multi-soliton representation of a growing interface.

Imposing periodic boundary conditions it is easily seen by inspection that the propagation of solitons lead to a growing interface. A particular simple

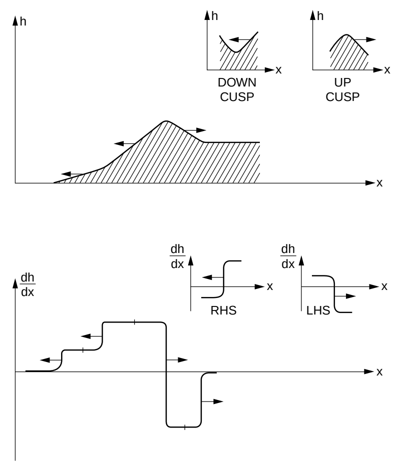

Figure 3: We depict the single cusp and soliton modes and a multi soliton representation of a growing interface. We show both the height profile and the slope profile.

growth mode corresponds to the propagation of a pair of matched right and left hand solitons. This mode is equivalent to the addition of layer upon layer in the evolution of the height profile. In Fig. 3 we have shown the single cusps and solitons and a multi soliton representation of a growing interface

The WKB approach also allows us to associate a dynamics with the propagation of growth modes. Only the noise-induced left hand solitons are endowed with dynamical properties. In terms of the boundary values u_+ and u_- we obtain the Galilean invariant action $S = (1/6)\lambda\nu|u_+ - u_-|^3 t$, from the Hamiltonian H the energy $E = (2/3)\lambda\nu(u_+^3 - u_-^3)$, and from $\Pi = \int dx h\nabla p$ the momentum $\Pi = \nu(u_+^2 - u_-^2)$. Considering specifically a two-soliton mode composed of a right hand and left hand soliton with amplitude u and with vanishing boundary conditions it carries action $S = (1/6)|u|^3 t$, energy $E = -(2/3)\lambda\nu|u|^3$, momentum $\Pi = \nu|u|u$ and propagates with velocity $v = -\lambda u/2$. Eliminating the velocity we obtain the soliton dispersion law and from (5) the pair soliton distribution, x denotes the center of mass coordinate,

$$E = -\frac{4}{3}\frac{\lambda}{\nu^{1/2}}|\Pi|^{3/2} \quad \text{and} \quad P(x,t) \propto \exp\left(-\frac{4}{3}\frac{\nu}{\Delta\lambda^2}\frac{x^3}{t^2}\right) . \tag{10}$$

From these two expressions we draw two results: i) It follows easily from the spectral properties of the height correlations that the exponent 3/2 in the soliton pair dispersion law can be identified with the dynamic exponents z. The dynamical scaling is thus associated with the low lying gapless soliton modes. ii) In the stochastic growth mode characterized by a gas of independent pair solitons the solitons perform random walk with the characteristics of superdiffusion, e.g., the distribution implies the mean square displacement $\langle x^2\rangle(t) \propto (\Delta\lambda^2/\nu)^{1/z} t^{2/z}$ with $z = 3/2$.

4 Towards higher dimensions

In dimensions beyond $d = 1$ the scaling and morphology are more involved. From the DRG and the mapping to directed polymers if follows that $d = 2$ is a lower critical dimension. Above $d = 2$ the system undergoes a kinetic phase transition at a finite coupling from a weak coupling smooth phase with exponents $z = 2$ and $\zeta = 0$ to a strong coupling rough phase. An operator expansion method yields $z = 4d/(2d + 1)$ for $d > 1$ [9]. Two other heuristics proposals are $z = 2(d + 2)/(d + 3)$ [10] and $z = (2d + 1)/(d + 1)$ [11] based on numerics; they both yield $z = 3/2$ in $d = 1$. Common to all these expressions is that $z \to 2$ for $d \to \infty$, i.e., yielding an infinite upper critical dimension. [5].

Within the present WKB approach the evolution of the morphology of the growing interface is given by the appropriate strong coupling solutions of the "classical" equations (7) and (8). In one dimension we were able analytically to find soliton solutions and thus "parameterize" a growing interface in terms of a gas of kinematically matched solitons. We have also analyzed the equations numerically in one dimension but here the negative diffusion coefficient in the equation for the noise field p renders the coupled equations numerically unstable and we were only able to verify certain soliton collision configurations. The same instability problem will remain in higher dimension and it is at the moment not clear how to find solutions numerically.

Starting from random initial conditions the time evolution of the noiseless Burgers equation is damped [12]. Since the nonlinear Cole-Hopf transformation [1] relates the slope field to a linear diffusive field the transient morphology can be analyzed by saddle point methods and is characterized by a gas of localized right hand solitons connected by ramp solutions; for the height field this pattern formation corresponds to a gas of downward cusps connected by parabolic segments. In two and higher dimensions one finds a corresponding morphology [12]. The height field forms a damped cellular network of growing "domes" connected by "valley's". The domes and valleys correspond to the parabolic segments and downward cusps in one dimension, respectively.

When the noise is "turned on" the right hand soliton in one dimension is supplemented by a noise induced left hand soliton in the WKB interpretation and as discussed above the morphology can be discussed in terms of connected right hand and left hand solitons. For the height field one version of this morphology corresponds to growing and decaying plateaus. The moving steps connecting the plateaus correspond to soliton pairs in the slope field. Similarly, we believe that a corresponding morphology takes place in higher dimensions. In for example two dimensions the stationary growth morphology in the height field is composed of growing and decaying islands forming a cellular network. The islands are connected by ramps or steps propagating with a velocity dictated by the Galilean invariance of the corresponding Burgers equation. The statistical weight of a morphology is then given by the action associated with a given time evolution from h_1 to h in a time t. If we parameterize the islands by linear segments and ignore vertex corrections we can make use of the one dimensional soliton solutions and connect the island by means of pair-solitons of amplitude

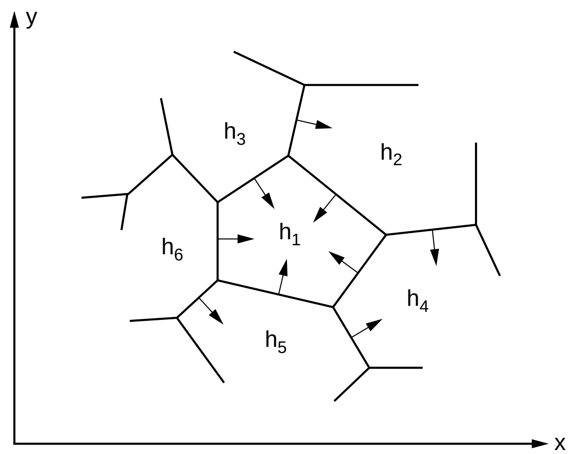

Figure 4: We depict the cellular network of growing and decaying islands in the height profile. The network is formed of domain walls or solitons constituting the fundamental growth modes.

u. The steps will then propagate with a velocity v or order λu and carry an action of order $\lambda\nu u^3 t$ per. unit length. In Fig. 4 we have depicted a growth configuration for the height field in two dimensions.

We now have enough elements of a future more detailed theory to venture a proposal for the dynamic exponent z. First we notice that the localized growth modes lie on the stationary manifold $H = 0$. In one dimension this manifold is given by $p = -2\nu\nabla^2 h$, yielding the action $S \propto \lambda\nu|u|^3 t$ and the stationary distribution $P_{\mathrm{st}}(h) \propto \exp[-(\nu/\Delta)\int dx(dh/dx)^2]$. In higher dimensions the stationary manifold is unknown and the stationary distribution is expected to be non-Gaussian and skew [6]. However, assuming that the Galilee invariant action for a step in h is a function of the slope change δu (ignoring the vector character of $\vec{u}$) and expressing S in the form $S \propto \lambda\nu|u|^\alpha \ell^{d-1} t$, where the factor ℓ^{d-1} arises from the extended character of the higher dimensional "solitons" or "domain walls" providing the growth modes, a natural choice interpolating between $d = 1$ and $d = \infty$ is to set $\alpha = d + 2$. Finally, noting that $\delta u \propto v$ and balancing the space and time dependencies of v and ℓ we infer a dynamic exponent in agreement with Wolf and Kertez [11]

$$z = \frac{2d+1}{d+1} \tag{11}$$

This expression agrees with $z = 3/2$ in $d = 1$. In $d = 2$ we obtain $z = 5/3$ which is close to the numerical value [1]. For $d \to \infty$ we have $z \to 2$ and the analysis yields an infinite upper critical dimension.

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References

[1] A. L. Barabasi and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995); T. Halpin-Healy and Y. C. Zhang, Phys. Rep. **254**, 215 (1995); J. Krug, Adv. Phys. **46**, 139 (1997).

[2] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A **39**, 3053 (1989).

[3] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. **36**, 867 (1976); D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[4] E. Frey and U. C. Täuber, Phys. Rev. E **50**, 1024 (1994); E. Frey, U. C. Täuber, and T. Hwa, Phys. Rev. E **53**, 4424 (1996); K. J. Wiese, Phys. Rev. E **56**, 5013 (1997); K. J. Wiese, J. Stat. Phys. **93**, 143 (1998).

[5] T. Hwa and E. Frey, Phys. Rev. A **44**, R7873 (1991); F. Colaiori and M. A. Moore, Phys. Rev. Lett. **86**, 3946 (2001); F. Colaiori and M. A. Moore, Phys. Rev. E **63**, 057103 (2001); C. Castellano *et al.*, Phys. Rev. E **59**, 6460 (1999).

[6] B. Derrida and J. L. Lebowitz, Phys. Rev. Lett. **80**, 209 (1998); B. Derrida and C. Appert, J. Stat. Phys. **94**, 1 (1999).

[7] M. Prähofer and H. Spohn, Phys. Rev. Lett. **84**, 4882 (2000);

[8] H. C. Fogedby, A. B. Eriksson, and L. V. Mikheev, Phys. Rev. Lett. **75**, 1883 (1995); H. C. Fogedby, Phys. Rev. E **59**, 5065 (1999); H. C. Fogedby, Europhys. Lett **56**, 492 (2001); H. C. Fogedby, Eur. Phys. J. B **20**, 153 (2001); H. C. Fogedby and A. Brandenburg, (2001), cond-mat/0105100.

[9] M. Lässig, Phys. Rev. Lett. **80**, 2366 (1998).

[10] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).

[11] D. E. Wolf and J. Kertész, Europhys. Lett **4**, 651 (1997).

[12] W. A. Woyczynski, *Burgers–KPZ Turbulence* (Springer–Verlag, New York, 1998).